Spontaneous Symmetry Breaking

- Peskin 11.1, 20.1, 21.2
- Schwartz 28

Symmetries play a central role in QFT and in our understanding of Nature. They are a property of the <u>dynamics</u>, that constrain and determine the spectrum and physical processes.

Sometimes, these symmetries are "hidden" or "spontaneously broken". We say that the vacuum does not preserve the symmetry.

It is even fair to say that spontaneously broken symmetries are even more powerful than regular symmetries, since they constrain both the interactions & the matter content at low energies. This has to do with the Goldstone modes. For the moment, we will discuss a few examples of SSB.

■ $\mathbb{Z}_2$

Start with a real scalar field $\phi(x)$,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi)$$

which has a $\mathbb{Z}_2$ symm

$$\phi(x) \xrightarrow{\{1,-1\}} -\phi(x)$$

The most general choice with $d \leq 4$ is

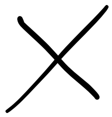
$$V(\phi) = \frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4}\phi^4$$

Note that, by symmetry, a term like

$$\frac{\tilde{\lambda}}{3}\phi^3$$

is forbidden, $\tilde{\lambda} = 0$.

Consider $\mu^2 > 0$. The theory has one spin zero particle with mass $m_\phi = \mu$, with int.

 $= -6i\lambda$

The interaction contains an <u>even</u> number of particles, so no diag. contains an odd number of ext. legs. For instance,

$$\mathcal{A}(\phi\phi \to \phi\phi\phi) = 0.$$

You can write this in a more pedantic way. A Z_2 operator acts on multiparticle states as

$$Z_2 |p_1 \cdots p_n\rangle = (-1)^n |p_1 \cdots p_n\rangle$$

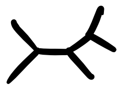
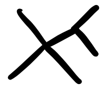
That Z_2 is a symmetry of the dynamics means $[Z_2, S] = 0$, so

$$S_{\alpha_n \to \beta_m} = \langle p_1 \cdots p_m \underset{\mathbb{1} = (Z_2)^2}{|}\, S \,\underset{\mathbb{1} = (Z_2)^2}{|} p_1 \cdots p_n \rangle$$

$$= (-1)^{n+m} S_{\alpha_n \to \beta_m}$$

So the S-matrix is block-diagonal with respect even/odd states.

- If Z_2 is broken, then $\tilde{\lambda} \neq 0$ and

$A(\phi\phi \to \phi\phi\phi) =$  $+$  $\neq 0$

So if $\tilde{\lambda} \neq 0$ the symmetry is explicitly broken.

If $\tilde{\lambda}/\sqrt{\mu} \sim \lambda$, we can never talk about Z_2 to begin with.

However, if $\tilde{\lambda}$ is small, then Z_2 is approximate.

Note that, doing experiments with finite precision, we will <u>never</u> know if a symmetry is exact or broken.

For instance, baryon number. Today it is a symmetry of Nature, but maybe tomorow someone sees a proton decay.

• Consider now a negative mass term,

$$V(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4}\phi^4$$

with $\mu^2 > 0$.

Quantizing the theory requires to find the field configuration that minimizes the energy, and study fluctuations around it.

An obvious solution is to take a constant field configuration at the minimum of the potential. Any spatial derivative contributes to $(\vec{\nabla}\phi)$.

By quantization we mean to include quantum effects on the degrees of freedom around the classical vacuum. So we write

$$\phi(x) = \phi_{min} + \chi(x)$$

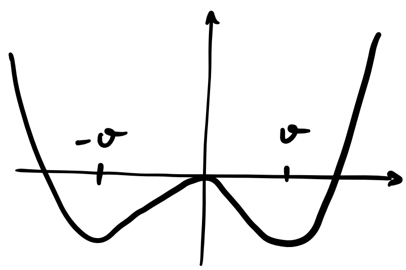
and study the dynamics of χ.

From the path integral,

$$e^{-\frac{1}{\hbar}S_E} = e^{-\frac{1}{\hbar}\int d^4x_E\left(\frac{1}{2}(\partial\phi)^2 + V(\phi)\right)}$$

the field configurations that dominate at $\hbar \to 0$ are those that minimize S_E, so $\phi = \phi_{min}$.
So we expand $S = S_0 + S_{int}$ around ϕ_{min}.

- For the real scalar field,



$$v \equiv \mu / \sqrt{\lambda}$$

we have $\langle \phi \rangle = \pm v \neq 0$.
Let's "pick" $\langle \phi \rangle = v$. Why not $-v$? We'll come back to it later.
Writing the potential as

$$V(\phi) = \frac{\lambda}{4}(\phi^2 - v^2)^2 - \frac{\mu^4}{4\lambda}$$

↳ "irrelevant" constant.

Then

$$V(\phi = v + \chi) = \frac{\lambda}{4}(v^2 + 2v\chi + \chi^2 - v^2)^2$$

$$= \lambda v^2 \chi^2 + \lambda v \chi^3 + \frac{\lambda}{4}\chi^4.$$

while

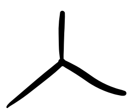
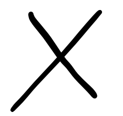
$$\frac{1}{2}(\partial_\mu \phi)^2 \to \frac{1}{2}(\partial_\mu \chi)^2 .$$

The theory contains a spin-zero particle χ with mass

$$m_\chi = \sqrt{2\lambda}\, v = \sqrt{2}\,\mu .$$

Before we had $m_\phi = \mu$, but the numerical factor is inconsequential.

· The interactions are

 $= -i6\lambda v$,  $= -i6\lambda$

The 4-pt vertex is still there. But now we have a trilinear one! We have

$$A(\chi\chi \to \chi\chi\chi) \neq 0 .$$

So, the Z_2 symmetry is "broken"?

The potential is the same for which we

argued that ϕ had a Z_2.

So, Z_2 is not "broken", it is just that the physical implications are different.

A clever experimentalist will realize that they can explain Z_2-preserving and Z_2-violating processes with the same parameter λ & m_χ.

So

$$\chi = 3\lambda v = \frac{3}{\sqrt{2}} \frac{m_\chi}{\sqrt{\lambda}}$$

is totally fixed by the Z_2-preserving terms.

• To be clear: the Lagrangian is still invariant under the Z_2. In terms of ϕ,

$$\phi \to -\phi$$

but in terms of χ, the symm is

$$v + \chi \to -v - \chi \Rightarrow \chi \to \chi' = -2v - \chi$$

This leaves $\mathcal{L}$ invariant.

So χ still transforms under Z_2. The point is that it transforms under a non-linear representation of Z_2.

This is the origin of the word "hidden". The symm. is still there, but the degrees of freedom transform non-linearly, making the consequences less transparent.

- Suppose we started with

$$\phi = -v + \chi'$$

we can do $\chi = -\chi'$, so $\phi = -v - \chi \underset{\text{by } Z_2}{=} v + \chi$

so is the same we did before.

Since V is Z_2-symm, the vacuum is Z_2-symm. and there is no difference between the minima we choose.

U(1)

Consider now a theory of a complex scalar field $\phi(x)$ with the global

$$\phi(x) \to e^{i\alpha}\phi(x)$$

The consequence of this is that charge is conserved.

$$Q|p_1 \cdots p_{n_+} \bar{p}_1 \cdots \bar{p}_{n_-}\rangle = (n_+ - n_-)|p_1 \cdots p_{n_+} \bar{p}_1 \cdots \bar{p}_{n_-}\rangle$$

Therefore, proceeding as before,

$$\langle k_1 \cdots k_{m_+} \bar{k}_1 \cdots \bar{k}_{-}| e^{i\alpha Q} e^{-i\alpha Q} S e^{i\alpha Q} e^{-i\alpha Q} |p_1 \cdots p_{n_+} \bar{p}_1 \cdots \bar{p}_{n_-}\rangle$$

$$= e^{-i\alpha[(n_+ - n_-) - (m_+ - m_-)]} \langle \cdots \rangle$$

$$\Rightarrow Q_{in} = n_+ - n_- = m_+ - m_- = Q_{out}$$

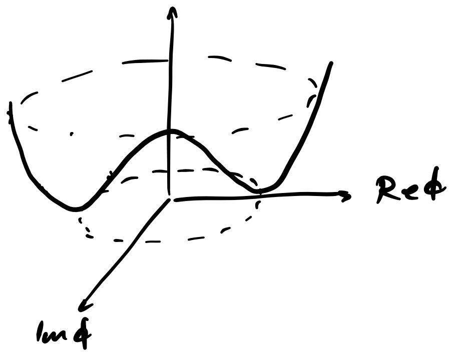
So the charge is conserved.

- Consider now the spontaneously broken theory.

$$V[\phi] = \lambda\left(|\phi|^2 - \frac{\mu^2}{2\lambda}\right) + \text{ctnt.}$$

With $v = \frac{\mu^2}{\lambda}$, there is a family of minima:

$$\langle \phi \rangle = \phi_{min} = \frac{v}{\sqrt{2}} e^{i\theta}$$



There is a family of minima labelled by θ.

We choose $\langle \phi \rangle = \frac{v}{\sqrt{2}}$.

$$\phi(x) = \frac{v}{\sqrt{2}} + \chi(x)$$

with

$$\chi(x) = \frac{\sigma(x) + i\eta(x)}{\sqrt{2}}$$

σ & η are two degrees of freedom fully describing the two d.o.f. of ϕ.

The Lagrangian is

$$\partial_\mu \phi^* \partial^\mu \phi = \frac{1}{2}(\partial_\mu \eta)^2 + \frac{1}{2}(\partial \sigma)^2$$

$$V(\phi) = \lambda v^2 \sigma^2 + \lambda v \sigma (\sigma^2 + \eta^2) + \frac{\lambda}{4}(\sigma^2 + \eta^2)^2$$

So, we have

- One massless field, η. $m_\eta = 0$.
- One massive, σ, $m_\sigma = \sqrt{2\lambda}\, v = \sqrt{2}\,\mu$.
- A bunch of interactions, all determined in terms of μ & λ.
- No conserved charge.

The presence of the massless η is dictated by symmetry.

Fluctuations of $\mathrm{Im}\,\phi$ have zero energy if $\eta \sim$ constant.

This is because they correspond to a global symm:

$$\phi(x) \to \phi(x) + i\alpha\,\phi(x)$$

$$\phi = v \qquad\qquad \to \quad \eta = \alpha v \sim \text{global transf.}$$

$$v \to v + i\alpha v$$

$$E(\vec{k}) \xrightarrow{\vec{k}\to 0} 0 \quad \Rightarrow \quad \eta \text{ is massless}$$

We shall see in the next lecture that the masslessness is very general feature.

- Again, the symmetry is still there, but "hidden". The implications of the symmetry is not only that two param. control all couplings, but that there is a massless mode!

Take the action of the $U(1)$:

$$\phi \to \phi' = \frac{v}{\sqrt{2}} + \chi' = e^{i\alpha}\phi = e^{i\alpha}\left(\frac{v}{\sqrt{2}} + \chi\right)$$

$$\hookrightarrow \chi' = e^{i\alpha}\chi + \frac{v}{\sqrt{2}}\left(e^{i\alpha} - 1\right)$$

Again, χ' transforms non-linearly. Note that χ^2 breaks this "shift" symmetry.

- One may use polar coordinates in field space.

$$\phi(x) = \frac{v + \rho(x)}{\sqrt{2}} e^{i\frac{\theta(x)}{v}}$$

with $\langle\rho\rangle = \langle\theta\rangle = 0$.

The symmetry acts on them as

$$\rho \to \rho$$

$$\theta \to \theta + \alpha v$$

Not only the shift symm. does not allow θ to have any mass, but it forbids any potential.

The Lagrangian:

$$\partial_\mu \phi^* \partial^\mu \phi = \frac{1}{2} \left| \partial_\mu \rho + (v+\rho) \frac{-i\partial_\mu \theta}{v} \right|^2$$

$$= \frac{1}{2} (\partial \rho)^2 + \frac{1}{2} \left(1 + \frac{\rho}{v}\right)^2 (\partial \theta)^2$$

$$= \frac{1}{2} (\partial \rho)^2 + \frac{1}{2} (\partial \theta)^2 + \frac{\rho}{v} (\partial \theta)^2 + \frac{\rho^2}{2v^2} (\partial \theta)^2$$

$$V[\phi] = \lambda v^2 \rho^2 + \lambda v \rho^3 + \frac{\lambda}{4} \rho^4$$

- θ disappears from the potential. It has only derivative interactions.
- The theory has $d=5$, $d=6$ interactions. Is however still renormalizable, equivalent to the ϕ theory.
- Derivative interactions controlled by the scale of SSB.

- SO(N)

It is clear that we can generalize the SSB of U(1) to SO(N) by considering

$$\phi(x) = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_N \end{pmatrix}$$

and

$$\mathcal{L} = \frac{1}{2}\partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi} - V(\phi)$$

$$; \; V(\phi) = -\frac{1}{2}\mu^2 \underbrace{\phi^2}_{\hookrightarrow \phi\cdot\phi} + \frac{\lambda}{4} \underbrace{\phi^4}_{\hookrightarrow (\phi\cdot\phi)^2}$$

The potential is equivalent to

$$V(\phi) = \frac{\lambda}{4}(\phi^2 - v^2)^2 \quad ; \; v^2 = \mu^2/\lambda$$

The vacua are defined by

$$\langle\phi\rangle^2 = v^2$$

which is the equation of an (N-1) sphere.

Since all vacua are equivalent, like before, with full generality we can pick

$$\langle \phi \rangle = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ v \end{pmatrix}$$

This "breaks" the SO(N) symmetry down to SO(N-1).

Fluctuations around this minimum can be param. as

$$\phi(x) = \begin{pmatrix} \pi_1(x) \\ \vdots \\ \pi_{N-1}(x) \\ v + \rho(x) \end{pmatrix}$$

So

$$\mathcal{L} = \frac{1}{2}(\partial\rho)^2 + \frac{1}{2}(\partial\pi_i)^2 - V(\pi,\rho)$$

$$; \; V = \frac{1}{2} m_\rho^2 \rho^2 + \lambda v(\rho^2 + \pi_i^2)\rho + \frac{\lambda}{4}(\rho^2 + \pi_i^2)^2$$

There is a single massive mode, ρ, and N-1 massless modes, the goldstones.

As you see, we have these massless modes every time we have SSB of continous symmetries. This will be the focus of next lecture.

■ Appendix: Solitons in the Z_2 theory

There is an interesting phenomenon in the Z_2 theory.

SSB exists in QFT only because we deal with infinitely many d.o.f. In QM, the vacuum is in a superposition $|0_+\rangle \pm |0_-\rangle$, which is Z_2 symm. In QFT, $\langle 0_+|H|0_-\rangle = 0$, so $|0_+\rangle$ & $|0_-\rangle$ are exactly degenerate. Under an external perturbation, one selects one vacuum instead of the other, and one has SSB. See Weinberg ch 19 for a detailed discussion. Probably you'll see it in Advanced QFT.

Bottom line: some parts of the universe might end up at v, some at $-v$. What happens in the middle?

The Lagrangian has static (time-independent) solutions to the e.o.m.

For a static solution:

$$-\delta \mathcal{L} = \delta \int dx \left[\frac{1}{2}(\partial_x \phi)^2 + V(\phi) \right] = 0$$

Mathematically, this is a particle, with coord. ϕ & time x in a potential $-V(x)$:

$$\delta \int dt\, \mathcal{L}' = \delta \int dt \left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + V(x) \right] = 0$$

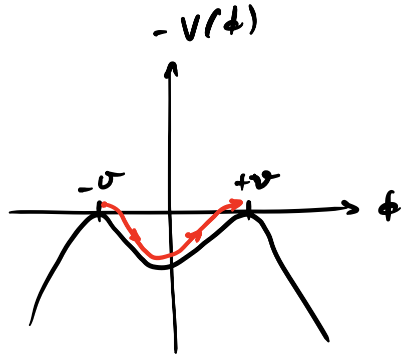
Any trajectory corresponds to a sol of the field equation.

To get a finite energy sol., ϕ must be at a min of $V(\phi)$ as $x \to \pm\infty$.

For the particle, it must be at zeros of potential at $t \to \pm\infty$. So, at $x = \pm v$.

If it sits at $x = \pm v$, solution.

Non-trivial traj.

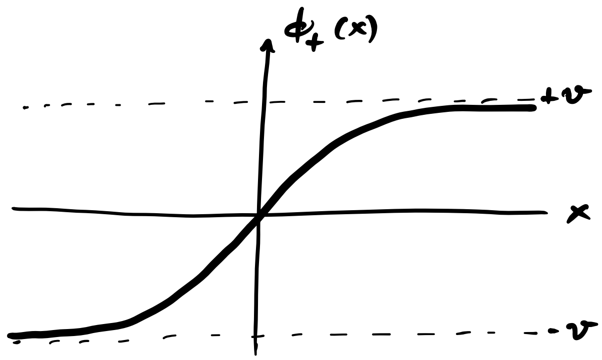


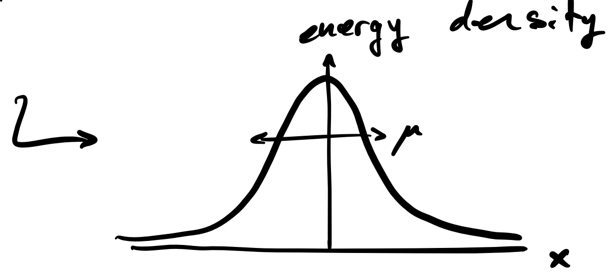
The traj is a sol of

$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 - V(x) = 0 \longrightarrow \frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 = V(\phi)$$

In our case,

$$\phi_\pm(x) = \pm v \tanh \mu x$$





So the field configuration interpolates between the two vacua.

In the x-direction, energy is localized, like a particle. In the y, z, however, it extends to infinity, and therefore it is a domain wall, separating two regions of space at diferent minima.